COMPOSITES IN CONSTRUCTIONS

Proceedings of the International Conference Composites in Constructions - CCC2003
Cosenza, Italy, September 16-19, 2003

Edited by

Domenico Bruno
University of Calabria, Italy

Giuseppe Spadea
University of Calabria, Italy

Ramnath Narayan Swamy
University of Sheffield, UK

EDITORIALE BIOS

Composites in Constructions, Bruno et al (eds), ©2003, Editoriale Bios

Stress-strain relationship of damaged solids containing elliptic cracks

Huang Hsing Pan
Kaohsiung University of Applied Sciences, Kaohsiung, Taiwan

ABSTRACT: A mean-field approach different from the self-consistent model is used to simulate the stress-strain curve of a damaged solid with elliptic cracks. The effective secant moduli of cracked materials strongly depend on the Poisson ratio of the matrix and the crack density, and only the shear modulus displays a certain degree of sensitivity on the crack shapes The results also show that the overall stress-strain curves of the cracked body affected by the crack shapes are found to be weak. Meanwhile, the result of circular cracks can represent the overall dilatational behavior. The effective secant moduli of damaged solids are also explicitly established. As compared with the experimental data of cement paste and mortar, the derived theory is in an acceptable range of accuracy.

1 INTRODUCTION

The full stress-strain curve of the cement-matrix composite in general is nonlinear, and controlled by the nonlinear behavior of cement paste in its own right and the volume fraction. Many literatures (Wang et al.1978, Carreira & Chu 1985, Harah et al. 1990, Wee et al. 1996, Attard & Setunge 1996) have been presented to descript the stress-strain behavior of concrete composite. Besides, microcracking of cement paste, mortar or concrete is correlated with applied strain under uniaxial compression (Carrasquillo et al. 1981, Attiogbe & Darwin 1987, 1988). The nonlinear response of the damaged body is due to non-linearity of the cement or mortar as well as the cracks themselves. This leads to more nonlinear softening cement or mortar.

The aim of this study is to evaluate the stress-strain relationship of a cracked solid containing elliptic cracks theoretically. Assuming that the damaged material is a two-phase composite with homogeneously dispersed, elliptic cracks, that is, it consists of cement paste (or mortar, concrete) as the matrix and the elliptic cracks as the inclusions. Then, the composite as a whole is isotropic. The aspect ratio of elliptic crack γ is equal to w/l, where w = the width and l = the length of the crack, respectively. The two extreme shapes of the elliptic crack – circular crack and ribbon-shape crack – are represented by $\gamma=1$ and $\gamma \rightarrow 0$, in turns. The derived theory in part is based on the randomly oriented mean-field approach (Tandon & Weng 1986), and now extends it to the cracked body by using the crack density instead of the volume fraction of the inclusions. With the aid of the secant moduli of the matrix, the overall stress-strain curve of the cracked body is finally determined.

2 EFFECTIVE SECANT MODULI

To find the stress-strain curve of the cracked body, one needs to know the effective elastic moduli of the two-phase composite first. The mean-field approach (Mori & Tanaka 1973) instead of the self-consistent method (Law & Brockenbrough 1987) is used to determine the effective moduli of the composite. Detailed exposition of this method based on Eshelby's equivalent transformation strain ε^* (Eshelby 1957) can be found in Weng (1984). In the two-phase system, the matrix will be referred to as phase 0, and the cracks as phase 1. The volume concentration of the r-th phase will be denoted by c_r. Meanwhile, the elastic bulk and shear moduli of the r-th phase will be written as κ_r and μ_r respectively.

The M-T method calls for the introduction of a homogeneous comparison material with the moduli of the matrix, and subject it to boundary tractions giving rise to a uniform strain ε^0. Then, the overall strain of the composite material $\bar{\varepsilon}$, which is also the overall strain of the comparison material with filled ε^* (or eigenstrain), can be written as

$$\bar{\varepsilon} = \varepsilon^0 + c_1 < \varepsilon^* > \qquad (1)$$

where the angle brackets <□> = orientational average of the said quantity. The equivalent transformed strain depending on the uniform strain ε^0 can be expressed by

$$<\varepsilon^*>= A^*\varepsilon^0 \tag{2}$$

where A^* = eigenstrain concentration tensor. The components of the eigenstrain concentration tensor are dependent on the specific configuration of the inclusion arrangements (Pan & Weng 1995). By substituting (2) into (1), the effective elastic bulk and shear moduli of the isotropic composite, κ and μ, are found to be

$$\frac{\kappa}{\kappa_0}=\frac{1}{1+c_1 p} \tag{3}$$

$$\frac{\mu}{\mu_0}=\frac{1}{1+c_1 q} \tag{4}$$

where p and q are two constants in terms of the inclusion shape, the volume concentration and material properties of the constituents.

For the cracked body, the volume fraction of the inclusion c_1 will go to zero or $c_1 \to 0$. Then, one should replace the volume concentration c_1 by the crack density (Zhao et al. 1989) with the definition (Budiansky & O'Connell 1976) as

$$\eta=\frac{2N}{\pi}\frac{A^2}{P} \tag{5}$$

where η = crack density; N = total number of cracks per unit volume; A = area of the crack; and P = perimeter of the crack. The constants in (3) and (4) for the cracked material with 3-D randomly oriented elliptic cracks are simply found as

$$c_1 p=\frac{16}{9}\frac{1-\nu_0^2}{1-2\nu_0}\eta \tag{6}$$

$$\begin{aligned} c_1 q = \{&8(1-\nu_0)[E^2(1-\gamma^2)^2(10-7\nu_0) \\ &+4\nu_0(F-E)(E-F\gamma^2)\gamma^2]\eta\} \\ &/\{45[E^2(1-\gamma^2)^2(1-\nu_0) \\ &+\nu_0^2(F-E)(E-F\gamma^2)\gamma^2]\} \end{aligned} \tag{7}$$

where E = complete elliptic integral of the second kind, not the effective Young modulus; F = complete elliptic integral of the first kind; and ν_0 = Poisson ratio of the matrix. These integrals E and F depend on the crack shape γ. By substituting (6) into (3), the explicit form of the effective bulk modulus can be recast as

$$\frac{\kappa}{\kappa_0}=\frac{1}{1+\frac{16}{9}\frac{1-\nu_0^2}{1-2\nu_0}\eta} \tag{8}$$

Obviously, the effective bulk modulus is independent of the crack shape.

For circular cracks, $E = F = \pi/2$, and for the ribbon-shaped cracks, $E = 1$ and $F \to \infty$. Thereby, one can find the effective shear modulus of damaged material with circular cracks to be

$$\frac{\mu}{\mu_0}=\frac{1}{1+\frac{32}{45}\frac{(1-\nu_0)(5-\nu_0)}{2-\nu_0}\eta} \tag{9}$$

and that with ribbon-shaped cracks to be

$$\frac{\mu}{\mu_0}=\frac{1}{1+\frac{8}{45}(10-7\nu_0)\eta} \tag{10}$$

From (8)–(10), the crack density and the Poisson ratio of the matrix dominate the behavior of the cracked solid. It is noted that the effective moduli of the cracked solid in (8)-(10) is only good for the elastic behavior.

As mentioned, the stress-strain curve of the cracked materials is always nonlinear, and the effective moduli should not be constant under loading. This is because the elastic constraints of the matrix become weak as applied stresses or strains grow. Due to this reason, the concept of the secant moduli successfully predicted in the overall elastoplastic stress-strain relations of dual-phase metals (Weng 1990) is used.

To obtain an estimate of the stress-strain response of the cracked material, the analysis procedures allow the elastic moduli of the matrix to change. The explicit forms of the effective moduli of the damaged solid, then, change from elastic moduli to secant ones with the superscript "s" referred as the secant moduli, or the secant bulk modulus κ^s is of the form

$$\frac{\kappa^s}{\kappa_0^s}=\frac{1}{1+\frac{16}{9}\frac{1-\nu_0^{s\,2}}{1-2\nu_0^s}\eta} \tag{11}$$

and the effective secant shear modulus μ^s

$$\frac{\mu^s}{\mu_0^s}=\frac{1}{1+\frac{32}{45}\frac{(1-\nu_0^s)(5-\nu_0^s)}{2-\nu_0^s}\eta} \tag{12}$$

for the circular cracks, and

$$\frac{\mu^s}{\mu_0^s}=\frac{1}{1+\frac{8}{45}(10-7\nu_0^s)\eta} \tag{13}$$

for the ribbon-shaped cracks. In the meantime, the constant available to elliptic cracks in (7) also becomes

$$
\begin{aligned}
c_1 q = \{ & 8(1-\nu_0^s)[E^2(1-\gamma^2)^2(10-7\nu_0^s) \\
& +4\nu_0^s(F-E)(E-F\gamma^2)\gamma^2]\eta\} \\
/\{ & 45[E^2(1-\gamma^2)^2(1-\nu_0^s) \\
& +\nu_0^{s\,2}(F-E)(E-F\gamma^2)\gamma^2]\}
\end{aligned}
\tag{14}
$$

3 NUMERICAL RESULTS

It is now of interest to apply the developed theory to examine some practical material systems. Two materials - cement paste and mortar - are chosen to investigate the effects of microcracks on the stress-strain relation. The material properties of the constituents are shown in Table 1 for cement paste, and in Table 2 for mortar (Attiogbe & Darwin 1987, 1988). Both materials with water-cement ratios (w/c) of 0.5 were studied. At the beginning of the test, the applied strain $\varepsilon = 0$ and some pre-existing cracks exist in the matrix, for example, η=0.015 for cement paste in Table1.

Table 1. Material properties of cement paste with w/c=0.5.

Applied strain ε	Crack density η	E_0 (GPa)*	ν_0
0.0	0.015	17.50	0.240
0.0005	0.020	17.12	0.242
0.001	0.024	16.65	0.241
0.002	0.033	15.28	0.242
0.004	0.076	12.11	0.246
0.006	0.105	9.47	0.262

* E_0 = Young's modulus of the matrix.

Table 2. Material properties of mortar with w/c=0.5.

Applied strain ε	Crack density η	E_0 (GPa)	ν_0
0.0	0.008	33.00	0.200
0.0005	0.018	31.52	0.214
0.001	0.022	26.56	0.249
0.002	0.040	20.11	0.290
0.003	0.072	14.81	0.352
0.004	0.088	8.48	0.441

To simulate the stress-strain curves of cement paste and mortar, the data in Table 1 and Table 2 are used. When the applied strain increases, the stiffness moduli decrease. It allows the Young modulus and the Poisson ratio of the matrix in both tables to be taken as the secant ones. Besides, cement paste and mortar are damaged progressively and still can be assumed to remain isotropic during loading (Attiogbe & Darwin 1987).

The overall stress-strain curves of the damaged solid are determined from (11) – (14), and the relations – κ=E/3(1-2ν) and μ=E/2(1+ν) – of the independent constants are also used. Once the effective Young modulus $E^s(\varepsilon)$ of the cracked material is evaluated, the stress-strain curve is found by

$$\sigma = E^s(\varepsilon)\cdot\varepsilon \tag{15}$$

The calculated stress-strain curves for cement paste and mortar are depicted in Fig. 1 and Fig. 2 respectively. The elastic curve shown in both figures means the initial modulus of the matrix.

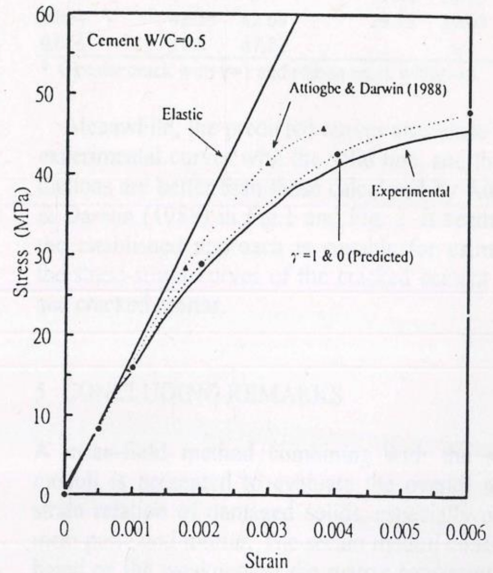

Figure 1. Stress-strain curves for cement paste with w/c=0.5.

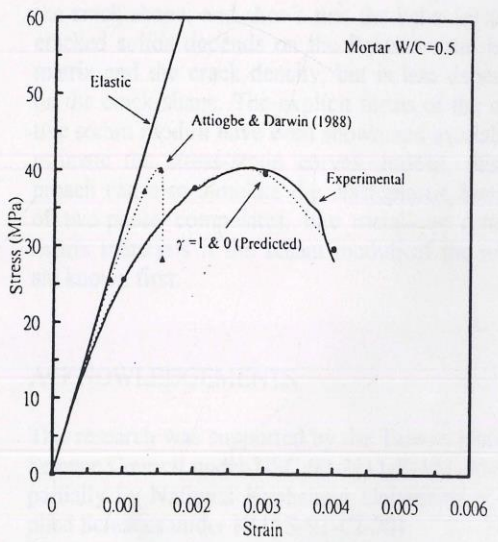

Figure 2. Stress-strain curves for mortar with w/c=0.5.

4 DISCUSSION

According to the derived secant moduli based on the mean-field approach and the experimental data, the results are discussed as follows.

4.1 *Effective moduli of the damaged solid*

The derived moduli of the cracked material in (11) and (14) indicate that the effective moduli strongly depend upon the secant Poisson ratio of the matrix and the crack density. The crack shape merely affects the shear modulus, but not the bulk one. As the shear moduli in (12) and (13) are calculated by substituting the Poisson ratio from 0 to 0.5, the modulus ratio varies from 1 to 12/13. This means the material containing circular cracks and that with ribbon-shape ones have the same shear modulus at $\nu_0 = 0$, and only 8% difference at $\nu_0 = 0.5$. In the other hand, the crack shape does not an important factor comparing with the Poisson ratio and the crack density. One might use the results of circulate cracks to represent the behavior of the cracked body.

The calculated moduli reduction affected by microcrack during applies strains are shown in Table 3 and Table 4 if the crack shape is circular. The effective Young modulus and Poisson ratio of the damaged material are denoted as E_c and ν_c respectively. From Table 3 and Table 4, the effective Young modulus reduces as applied strains increase for both cement paste and mortar. The effective Poisson ratio of cement paste in Table 3 does not change much at applied strain smaller than 0.004, but will increase as $\varepsilon > 0.004$. For the cracked mortar in Table 4, the effective Poisson ratio grows rapidly as ε increases. The tendency of ν_c is similar to ν_0.

Table 3. Moduli reduction of cement paste* with w/c=0.5

Applied strain	Young's modulus		Poisson' ratio	
ε	E_0 (GPa)	E_c (GPa)	ν_0	ν_c
0.0005	17.12	16.54	0.242	0.235
0.001	16.65	15.97	0.241	0.232
0.002	15.28	14.43	0.242	0.230
0.004	12.11	10.68	0.246	0.220
0.006	9.47	8.00	0.262	0.226

* Circular cracks included.

Table 4. Moduli reduction of mortar* with w/c=0.5

Applied strain	Young's modulus		Poisson' ratio	
ε	E_0 (GPa)	E_c (GPa)	ν_0	ν_c
0.0005	31.52	30.54	0.214	0.208
0.001	26.56	25.57	0.249	0.241
0.002	20.11	18.80	0.290	0.273
0.003	14.81	13.21	0.352	0.318
0.004	8.48	7.44	0.441	0.393

* Circular cracks included.

4.2 *Stress-strain relation*

The stress-strain curves are simulated and depicted in Fig.1 and Fig.2. The predicted curves are plotted in the dotted line with the points of the dark circle. The calculated stresses of cement paste and mortar are shown in Table 5.

Although the curve with circular cracks ($\gamma=1$) provides a little high value of stresses compared with that with the ribbon cracks ($\gamma \to 0$), they almost lie at the same curve. This result shows that the crack shape only display a little degree of sensitivity on the stress-strain curve.

Table 5. Crack shape effect on the stress–strain relation.

Applied strain	Cement paste (MPa)		Mortar (MPa)	
ε	$\gamma=1$*	$\gamma \to 0$*	$\gamma=1$	$\gamma \to 0$
0.0005	8.27	8.27	15.27	15.27
0.001	15.97	15.96	25.57	25.56
0.002	28.87	28.85	37.61	37.58
0.003	[illegible]	[illegible]	39062	39.53
0.004	42.72	42.68	29.75	29.61
0.006	47.8	47.92	[illegible]	[illegible]

* Circular crack with $\gamma=1$ and ribbon crack with $\gamma \to 0$.

Meanwhile, the predicted curves also close to the experimental curves with the solid line, and the predictions are better than those calculated by Attiogbe & Darwin (1988) in Fig.1 and Fig. 2. It seems that the established approach is suitable for estimating the stress-strain curves of the cracked cement paste and cracked mortar.

5 CONCLUDING REMARKS

A mean-field method combining with the secant moduli is presented to evaluate the overall stress-strain relation of damaged solids, especially on cement paste and mortar. The secant moduli concept is based on the weakness of the matrix constraints due to microcracking, and that reduces the effective moduli. This approach can account for the effect of the crack shape, and shows that the behavior of the cracked solids depends on the Poisson ratio of the matrix and the crack density, but is less dependent on the crack shape. The explicit forms of the effective secant moduli have been shown and available to estimate the stress-strain curves. Indeed, this approach can also simulate the elastoplastic behavior of two-phase composites, like metal- or ceramic-matrix materials if the secant moduli of the matrix are known first.

ACKNOWLEDGEMENTS

This research was supported by the Taiwan National Science Council under NSC 90-2211-E-151-004 and partially by National Kaohsiung University of Applied Sciences under KUAS-91-CI-001.

REFERENCES

Attard, M.M. & Setung, S. 1996. Stress-strain relationship of confined and unconfined concrete. *ACI Materials J.* 93(5): 432-442.

Attiogbe, E.K. & Darwin, D. 1987. Submicrocracking in cement paste and mortar. *ACI Materials J.* 84(6): 491-500.

Attiogbe, E.K. & Darwin, D. 1988. Strain due to submicrocracking in cement paste and mortar. *ACI Materials J.* 85(1): 3-11.

Budiansky, B. & O'Connell, R.J. 1976. Elastic moduli of a cracked solid. *Int. J. Solids Structures* 12: 81-97.

Carrasquillo, R.L., Slate, F.O. & Nilson, A.H. 1981. Microcracking and behavior of high strength concrete subject to short-term loading. *ACI J.* 78(3): 179-186.

Carreira, D.J. & Chu, K.H. 1985. Stress-strain relationship for plain concrete in compression. *ACI J.* 82: 797-804.

Eshelby, J.D. 1957. The determination of the elastic field of an ellipsoidal inclusion, and relation problem. *Proc. R. Soc. London Ser.* A241: 376-396.

Harsh, S., Shen, Z. & Darwin, D. 1990. Strain-rate sensitive behavior of cement paste and mortar in compression. *ACI Materials J.* 87(5): 508-515.

Law, N. & Brockenbrough, J.R. 1987. The effect of microcrack systems on the loss of stiffness of brittle solids. *Int. J. Solids Structures* 23(9): 1247-1268.

Mori, T. & Tanaka, K. 1973. Average stress in the matrix and average elastic energy of materials with misfitting inclusions. *Acta Metall.* 21: 571-574.

Pan, H.H. & Weng, G.J. 1995. Elastic moduli of heterogeneous solids with ellipsoidal inclusions and elliptic cracks. *Acta Mechanica* 110(1-4): 73-94.

Tandon, G.P. & Weng, G.J. 1986. Average stress in the matrix and effective moduli of randomly oriented composites. *Composite Science and Technology* 27: 111-132.

Wang, P.T., Shah, S.P. & Naaman, A.E. 1978. Stress-strain curves of normal and Lightweight concrete in compression. *ACI J.* 75: 603-611.

Wee, T.H., Chin, M.S. & Mansur, M.A. 1996. Stress-strain relationship of high-strength concrete in compression. *J. Materials in Civil Engineering* 8(2): 70-76.

Weng, G.J. 1984. Some elastic properties of reinforced solids, with special reference to isotropic ones containing spherical inclusion. *Int. J. Eng. Sci.* 22: 845-856.

Weng, G.J. 1990. The overall elastoplastic stress-strain relations of dual-phase metals. J. Mech. Phys. Solids 38: 419-441.

Zhao, Y.H., Tandon, G.P. & Weng, G.J. 1989. Elastic moduli for a class of porous materials. *Acta Mechanica* 76: 105-131.