MICROMECHANICS APPROACH FOR LONG-TERM STRESS-STRAIN RELATIONSHIPS OF CEMENT-MATRIX COMPOSITES

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Abstract

Based on a micromechanics-based theory and Burgers' rheological model, overall stress-strain relationships of the cement-matrix composites containing cement and fly ash/cement binders are determined for the material ages with 7 days, 14 days, 28 days, 2 months, 3 months, 6 months, 9 months, 12 months and 18 months. According to a four-parameter model, the stress-strain relation of the cement-based binder are established and simulated with respect to the material age, the strain rate and the size of the specimen. Four material parameters for two cement-based binders depending on the material age are also determined. The long-term stress-strain curves of mortars containing the volume concentrations c_1 =0.29, 0.38 and 0.49 are tested and also simulated by the inclusion method and the concept of secant moduli. The results show that the predicted stress-strain curves in the linear state are almost the same as the experimental ones, and the predicted curves in nonlinear part are slightly lower than, but close to, the experimental ones regardless of the volume concentration of inclusions and the material age. The proposed micromechanics model has the predictive power depending on the material ages for the stress-strain behaviour of a mortar that carries different from amount of aggregates.

Keywords: stress-strain, material age, micromechanics, cement-matrix composite

INTRODUCTION

Traditional models, based on phenomenological approach, have been proposed to estimate the stress-strain behaviour of the cement-matrix composite such as mortar or concrete. Most of them set the influence factors by means of the properties of the composite, such as the peak strength and the initial Young's modulus [1-4]. Most of them set the influence factors by means of the properties of the composite. The composite properties were not derived from the properties of the individual phases but rather by inverse fitting of the measures data. Thus the simulated stress-strain relations of the composite material are difficult to reproduce as the aggregate content or testing conditions changes, as in that case the material constants must be fitted again for a new composite system. Since concrete consists of the cement paste (or the binders) and the aggregates, its properties are highly dependent on the individual properties and volume concentration, and shape, of the aggregates. The approach here is developed and concentrates on long-term stress-strain behavior of cement-matrix composites so that it can be applied to conditions of different constitutions without having to determine the material properties of the composite for each and every case.

Based on a micromechanics-based approach, we take concrete as a composite with cement-based binders as the matrix and the aggregates as reinforcing inclusions. We also take the nonlinear response of the binders and the volume concentration of the aggregates as the two dominant factors in determining the overall response of the cement-based composites. The inclusions of the same shape but different sizes are taken to perfectly bonded to the matrix. The matrix phase here is to be broadly viewed as a combination of cement paste, fly ash, slag, silica fume, and other admixtures. The composite material as a whole then is a two-phase composite with an overall isotropy. Moreover the inclusions are taken to be homogeneously dispersed in the matrix, with an average shape grossly represented by a spheroidal inclusion with an aspect ratio α (the length-to-diameter ratio). In the two-phase system, the matrix will be referred to as phase 0, and the inclusions as phase 1. The volume concentration of the r-th phase will be denoted by c_r ($c_1 + c_0 = 1$), and the bulk and shear moduli of the r-th phase will be written as κ_r and μ_r, respectively.

Before we proceed to develop the composite theory, we will first characterize the nonlinear behavior of the matrix by experiments and then by theoretical simulations at different material ages. The latter will be accomplished based on Burgers' four-parameter rheological model. This would allow us to establish the secant moduli of the matrix at a given stage of deformation. The Eshelby-Mori-Tanaka theory [5-6] then will be invoked to determine the effective elastic moduli of the composite with randomly oriented ellipsoidal inclusions. Then following a previously established procedure [7-8] the linear relations will be extended to the nonlinear regime by means of the secant moduli. Unlike the elastic moduli, the secant moduli of the nonlinear binders and of the overall composite will continue to decrease during the course of deformation, and will have to be constantly updated in order to calculate the entire stress-strain curves. In this paper we shall focus on the stress-strain behavior of mortar at various material ages and concentrations of aggregates.

EXPERIMENTAL PROGRAM

Two cement-based binders and six mortars are tested and measured for their stress-strain curves, and each of them has nine material ages with 7 days, 14 days, 28 days, 2 months, 3 months, 6 months, 9 months, 12 months and 18 months. The water-to-binder ratio of cement binder and fly ash/cement binder is 0.45 and 0.41, respectively, and mortars were made from those two binders (matrix) but with different volume fractions, there are $c_1 = 0.29$, 0.38 and 0.49, of find aggregate (inclusions). The measured particle size of the inclusions was about 0.7~1.0mm, a specific gravity of 2.65, and a shape similar to that of a spheroid with an aspect ratio $\alpha = 1.13$. The elastic bulk modulus and shear modulus of the inclusion are $\kappa_1 = 19.46$ GPa and $\mu_1 = 18.44$ GPa. Mixture proportions of the cement-based composite are shown in Table 1 and Table 2, where the material at $c_1 = 0$ represents the cement-based binder.

Table 1: Mixture proportions of cement binder composite (kg/m^3)

c_1	water	cement	sand
0	586	1302	—
0.29	415	921	775
0.38	362	804	1014
0.49	298	623	1301

Table 2 Mixture proportions of fly ash/cement binder composite (kg/m^3)

c_1	water	cement	fly ash	sand
0	546	1133	200	—
0.29	384	796	140	788
0.38	334	693	123	1029
0.49	275	570	101	1316

Specimen size is of $10\phi\times20$ cm. At least six samples of each specimen were used to examine the material properties. Specimens were under a uniaxial compression with a constant strain rate $\dot{\varepsilon}=1\times10^{-5}$ /sec. The longitudinal and lateral strains were measured to plot the stress-strain curves, and to calculate the elastic moduli of the materials from ASTM C469.

BURGERS' FOUR-PARAMETER RHEOLOGICAL MODEL FOR THE CEMENT-BASED BINDER

The stress-strain relations of cement paste and the cement-based binders are nonlinear although its fracture behavior is brittle. The nonlinearity stems from the inelastic deformation of the binders itself as well as from possible microcracking under loadings [9]. Pan and Kuo [10] found that the nonlinear behavior of the cement-based binders can be represented by Burgers' four-parameter rheological model with two springs and two dashpots. Here, we continue to use Burgers' rheological model to simulate the stress-strain behavior of cement-based binders but by considering the effect of the material age as shown in Fig. 1, where two springs and two dashpots, depending on the material age denoted as *t*, are referred as $k_1(t)$, $k_2(t)$, $\eta_1(t)$ and $\eta_2(t)$ respectively. An external load $f(T)$ subjected to the binder, where *T* is the applied time, will take place the displacements u_1, u_2, and u_3 at k_1, η_1, and k_2 and η_2, respectively. Noting that the velocity $w=\dot{u}=\dot{u}_1+\dot{u}_2+\dot{u}_3$.

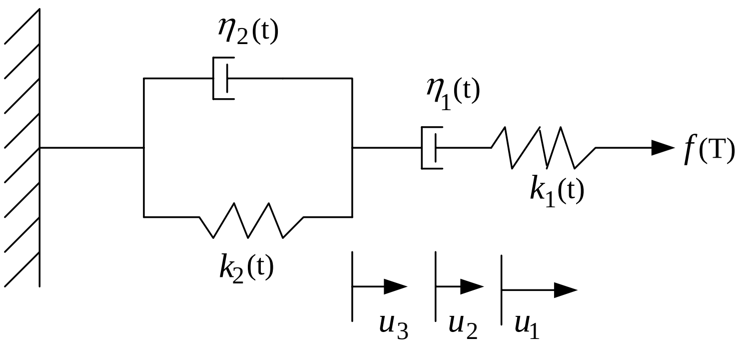

Figure 1: Burgers' model for cement-based binders depending on material age

From Fig.1, one readily arrives at the governing differential equation for this four-parameter model

$$\ddot{f}(T)+\left(\frac{k_1}{\eta_1}+\frac{k_1}{\eta_2}+\frac{k_2}{\eta_2}\right)\dot{f}(T)+\frac{k_1k_2}{\eta_1\eta_2}f(T)=k_1\dot{w}+\frac{k_1k_2}{\eta_2}w \tag{1}$$

Using a constant velocity ($\dot{w} = 0$) and the initial condition $f(0) = 0$, the applied load can be solved and expressed as

$$f(T) = b\left[e^{m_1 T} - \left(\frac{\eta_1 w + b}{b}\right)e^{m_2 T}\right] + \eta_1 w \tag{2}$$

where b is a material constant, whereas m_1 and m_2 are the characteristic roots. By interchanging the independent variable (method of parameter variables) related to the applied time and the applied strain, we can rewrite Eq. (2) to be of the general form

$$\frac{\sigma}{f_u} = 3.71\left[e^{m_1(\varepsilon\times 10^3)} - (1 + 0.27B)e^{m_2(\varepsilon\times 10^3)}\right] + B \tag{3}$$

with

$$m_1 = \frac{\sqrt{\{k_1(t)\cdot\eta_2(t) + [k_1(t) - k_2(t)]\cdot\eta_1(t)\}^2 + 4k_1(t)\cdot k_2(t)\cdot\eta_1^2(t)} - k_1(t)\cdot\eta_2(t) - [k_1(t) + k_2(t)]\cdot\eta_1(t)}{2\eta_1(t)\cdot\eta_2(t)}$$

$$m_2 = -\frac{\sqrt{\{k_1(t)\cdot\eta_2(t) + [k_1(t) - k_2(t)]\cdot\eta_1(t)\}^2 + 4k_1(t)\cdot k_2(t)\cdot\eta_1^2(t)} + k_1(t)\cdot\eta_2(t) + [k_1(t) + k_2(t)]\cdot\eta_1(t)}{2\eta_1(t)\cdot\eta_2(t)}$$

where σ, f_u, and ε are the stress, peak stress and strain of the cement-based binder, respectively. The unit of the stress, spring constants and dashpot constants is Pa, GN/m and GN-sec/m, respectively. Meanwhile, the parameter *B* in Eq. (3) has the relation as follows.

$$B = \eta_1(t)\cdot\frac{\dot{\varepsilon}}{A}\cdot\frac{L}{f_u} \tag{4}$$

where *A* and *L* are the area and the length of the specimen respectively, and the length unit is meter (m). Eqs. (3)-(4) show that the stress-strain relation of the cement-based binder depends on the applied strain rate, the size of specimen and the material age.

The comparisons between Eq. (3) and the experimental stress-strain curve at $c_1 = 0$ are shown in Figs. 2-3. One arrives at the four material parameters for the cement binder

$$k_1(t) = 0.05\cdot(\log t)^2 - 0.10\cdot\log t + 0.78\,,\quad r = 0.99 \tag{5}$$

$$k_2(t) = 0.02\cdot(\log t)^2 - 10.53\cdot\log t + 12.54\,,\quad r = 0.99 \tag{6}$$

$$\eta_1(t) = 0.13\cdot(\log t)^2 - 0.23\cdot\log t + 4.92\,,\quad r = 0.99 \tag{7}$$

$$\eta_2(t) = -3.14\cdot(\log t)^2 - 5.51\cdot\log t + 153.16\,,\quad r = 0.99 \tag{8}$$

and for the fly ash/cement binder

$$k_1(t) = 0.04 \cdot (\log t)^2 - 0.09 \cdot \log t + 0.77\,, \quad r = 0.99 \tag{9}$$

$$k_2(t) = 0.20 \cdot (\log t)^2 - 1.53 \cdot \log t + 12.35\,, \quad r = 0.99 \tag{10}$$

$$\eta_1(t) = 0.11 \cdot (\log t)^2 + 0.17 \cdot \log t + 4.90\,, \quad r = 0.99 \tag{11}$$

$$\eta_2(t) = -2.80 \cdot (\log t)^2 - 4.47 \cdot \log t + 148.03\,, \quad r = 0.99 \tag{12}$$

where the unit of the material age t is of day and these material parameters are suitable for $7 \le t \le 540$ (18 months), $\dot{\varepsilon} = 1 \times 10^{-5}$ /sec and specimen size $10\,\phi \times 20$ cm. The sample correlation coefficients r, in equations (5)-(12), all is close to 1 ($r = 0.99$). If the strain rate and the specimen size change, four material parameters will be certainly different. Figs. 2-3 show that the long-term nonlinear curves for two cement-based binders can be represented by the Burgers model.

In fact, this four-parameter model can also be used to simulate the mortar and concrete behavior [10]. The spring and dashpot constants for the cement-based binder, the mortar and concrete are of course different. Despite the good simulation such a curve-fitting procedure for the mortar and concrete has no predictive power and is what we have tried to avoid from the outset. This is what has prompted us to develop a micromechanics-based secant moduli method so that the long-term nonlinear behavior of cement-matrix composite (mortar or concrete) can be calculated from those of its cement-based binder at every volume fraction of the aggregates, c_1.

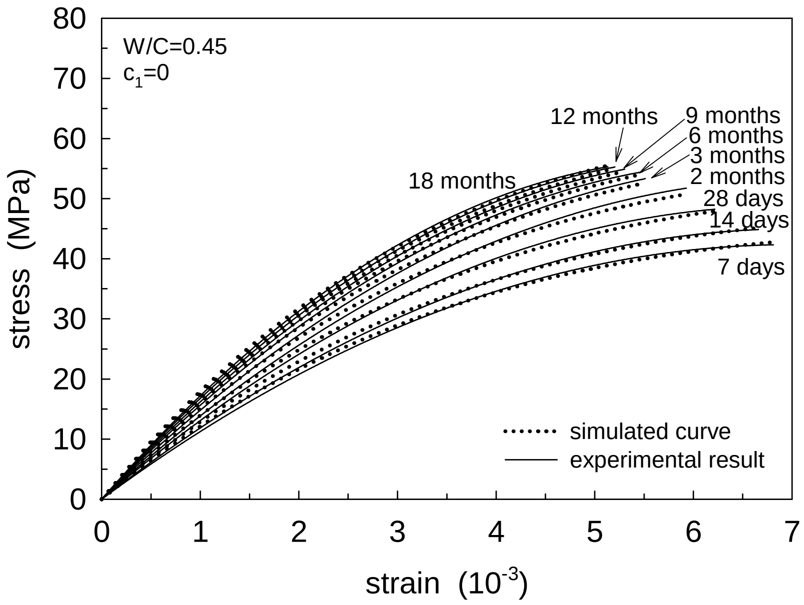

Figure 2: Simulations and experiments for cement binders ($c_1 = 0$)

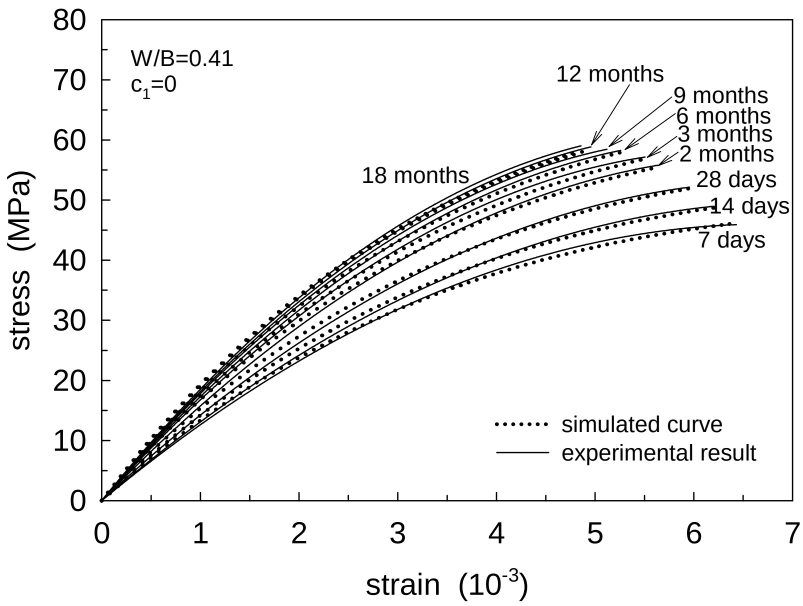

Figure 3: Simulations and experiments for fly ash/cement binders ($c_1 = 0$)

OVERALL STRESS-STRAIN CURVES OF CEMENT-MATRIX COMPOSITES

The Eshelby [5] and Mori-Tanaka [6] theory has their established applications to the determination of effective elastic moduli of a two-phase composite containing, aligned, two, or three-dimensional randomly oriented, ellipsoidal inclusions. Detailed exposition of this method can be found in Weng [11] for general theory with a special reference to spherical inclusions, and in Tandon and Weng [12] for both two and three-dimensional randomly oriented configuration. The calculated overall elastic moduli have been shown to be intimately related to the universal Hashin-Shtrikman bounds [13-14] for the isotropic case, and with Willis' anisotropic bounds [15-16] for the aligned configuration. The results of this method also coincide with those of the double-inclusion model when the double-cells have the same shape and orientation as the enclosed inclusions [17].

For the calculation of nonlinear stress-strain relations of the composite, we extend the elastic moduli to the secant moduli for the cement-based binder and the cement-matrix composite. This is the approach originally proposed by Tandon and Weng [7]. As secant moduli of the binder continues to decrease in the course of deformation, their values are smaller than their elastic counterparts, such as

$$\kappa_0^s \leq \kappa_0 \ \ ; \ \ \mu_0^s \leq \mu_0 \qquad (13)$$

where the superscript “s” stands for the secant moduli, which are the direct ratio of stress to the total strain. The secant bulk and shear moduli of the binder are related to its secant Young’s modulus E_0^s through

$$\kappa_0^s(\varepsilon) = \frac{E_0^s(\varepsilon)}{3(1-2\nu_0)} \quad ; \quad \mu_0^s(\varepsilon) = \frac{E_0^s(\varepsilon)}{2(1+\nu_0)} \tag{14}$$

where their dependence on the strain ε is now made explicit, and its Poisson’s ratio ν_0 is assumed to remain constant. It is noted that κ_0^s and μ_0^s are also material age-dependent. The secant Young modulus of the cement-based binder at a given stage of deformation can be determined from its basic definition

$$E_0^s(\varepsilon) = \sigma(\varepsilon)/\varepsilon \tag{15}$$

Here the cement-based binder (cement-matrix) composites, containing 3-D randomly oriented ellipsoidal inclusions (aggregates), are assumed to be isotropic. In the case that aggregates take an ideal spherical shape and are purely elastic, the overall effective secant bulk and shear moduli of the composite would simplify to [7,10,18]

$$\kappa^s = \frac{\kappa_0^s(3\kappa_1 + 4\mu_0^s) - 4c_1\mu_0^s(\kappa_0^s - \kappa_1)}{3\kappa_1 + 4\mu_0^s + 3c_1(\kappa_0^s - \kappa_1)} \tag{16}$$

$$\mu^s = \frac{\mu_0^s[5\mu_1(3\kappa_0^s + 4\mu_0^s) + c_0(\mu_0^s - \mu_1)(9\kappa_0^s + 8\mu_0^s)]}{5\mu_0^s(3\kappa_0^s + 4\mu_0^s) - 6c_0(\mu_0^s - \mu_1)(\kappa_0^s + 2\mu_0^s)} \tag{17}$$

where $c_0 = 1 - c_1$, is the volume fraction of the matrix (binder).

To briefly recapitulate, the stress-strain curve of the cement-matrix composite at a given volume fraction of the aggregates c_1 and material age *t*, is determined as follows. First, the stress-strain curve of the cement-based binders (without the aggregates) is obtained from the experiments. The curves are then simulated by Burgers model or Eq. (3), where four material parameters are determined listed in Eqs. (5)-(8) or (9)-(12). At this stage, the secant Young modulus $E_0^s(\varepsilon)$ can be calculated from Eq. (15), and so can the secant bulk and shear moduli of the binders from Eq. (14). The overall effective secant bulk and shear moduli of the composite (mortar or concrete), $\kappa^s(\varepsilon)$ and $\mu^s(\varepsilon)$, follow from Eqs. (16)-(17) if the average shape of the aggregate particles is spherical. Finally the overall effective secant Young modulus is obtained from

$$E^s(\varepsilon) = \frac{9\kappa^s(\varepsilon)\mu^s(\varepsilon)}{3\kappa^s(\varepsilon) + \mu^s(\varepsilon)} \tag{18}$$

Under a uniaxial compression, the overall stress-strain curve of the cement-matrix composite follows as

$$\overline{\sigma}(\varepsilon) = E^{s}(\varepsilon) \cdot \overline{\varepsilon} \tag{19}$$

which provides the desired overall stress *versus* strain ($\overline{\sigma}$ *vs.* $\overline{\varepsilon}$) relations at a specific material age.

RESULTS AND DISCUSSION

The binders and the mortars are applied to a uniaxial compression with a constant strain rate $\dot{\varepsilon} = 1\times10^{-5}\,/\mathrm{sec}$ and specimen size $10\phi\times20$ cm at different material ages. Then, the long-term stress-strain curves of two cement-based binders ($c_1 = 0$) can be calculated from Burgers' model combining with four age-dependent parameters, and the curves are shown in Figs. 2-3. Following the material properties of those two cement-based binders, the long-term stress-strain curves of mortars containing the volume concentrations c_1=0.29, 0.38 and 0.49 can be predicted by the developed micromechanics approach.

Figs. 4-6 are the long-term stress-strain curves of mortars with the same cement binder. In these figures the solid lines refer to the experimental curves and the dotted lines are the predicted ones. The results show that the simulated stress-strain curves in the elastic state are almost the same as the experimental curves, and the theoretical predictions in nonlinear part are slightly lower than, but close to, the experimental data regardless of the volume concentration of inclusions and the material age. This is because the aspect ratio of aggregates we choose in the theoretical calculations is an overall value $\alpha = 1.13$, and that may not coincide with the real shape of inclusions in the mortar. In addition, the defects (voids and microcracks) within the cement binder are not identical to those in the mortar.

The results for the mortar with fly ash/cement binder are also plotted in Figs. 7-9. The tendency of the stress-strain behavior between the theoretical predictions and the experiments is similar to the mortar with cement binder. It is observed from two cement-matrix composites that the predicted stress-strain curves are in close agreement with the test data. This agreement serves to substantiate the validity of the theory.

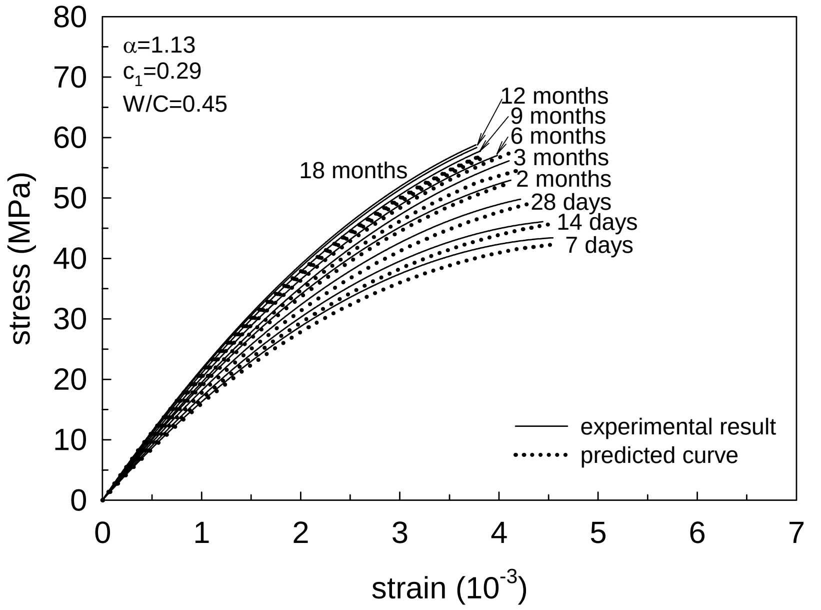

Figure 4: Long-term stress-strain curves of mortar at $c_1 = 0.29$ with cement binder

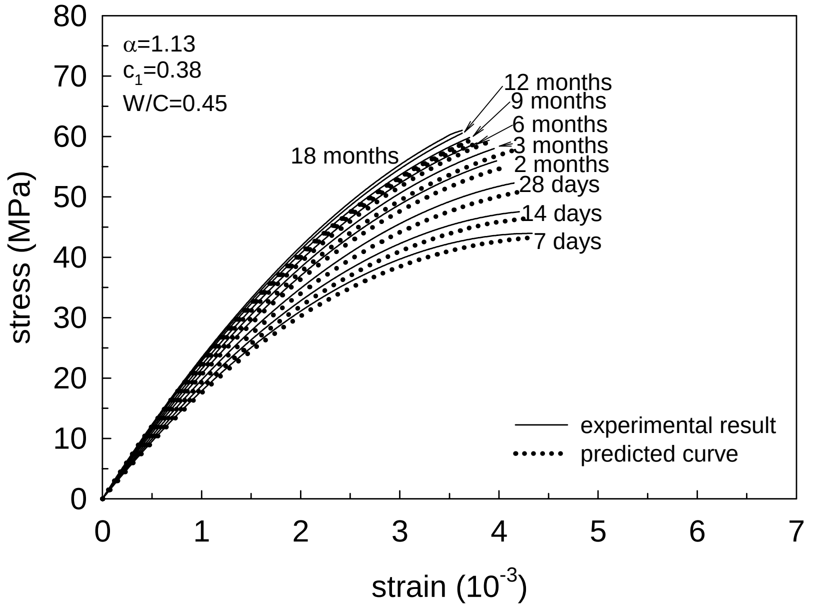

Figure 5: Long-term stress-strain curves of mortar at $c_1 = 0.38$ with cement binder

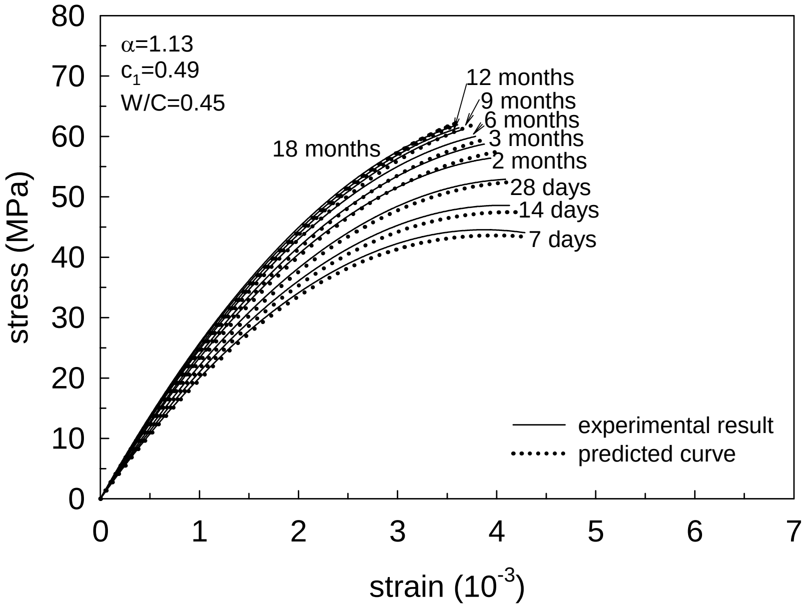

Figure 6: Long-term stress-strain curves of mortar at $c_1 = 0.49$ with cement binder

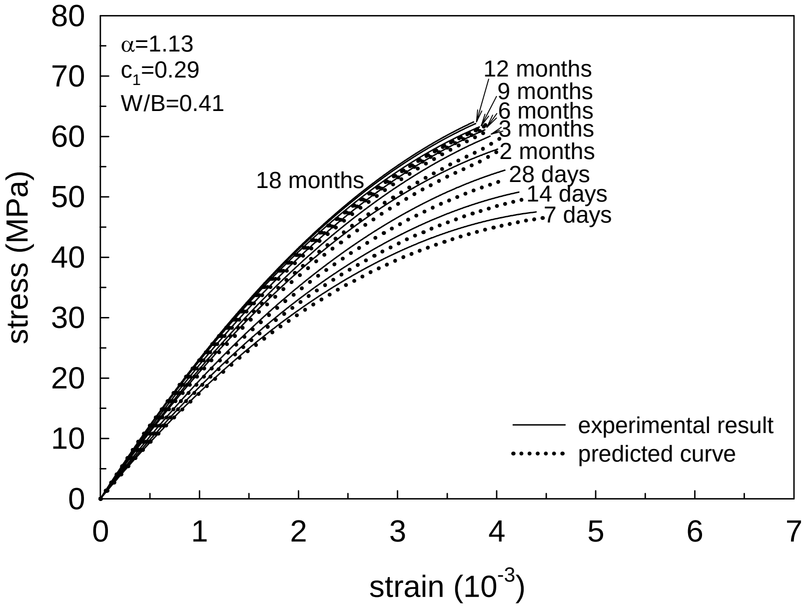

Figure 7: Long-term stress-strain curves of mortar at $c_1 = 0.29$ with fly ash/cement binder

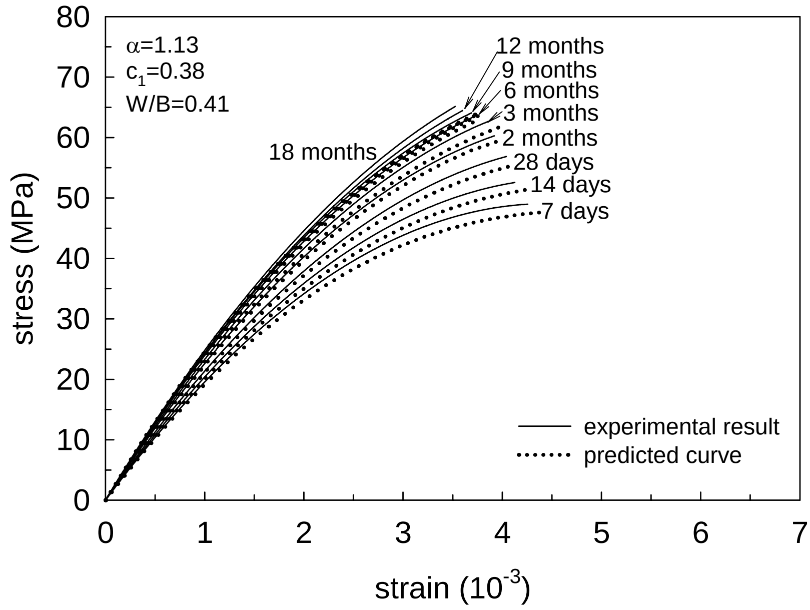

Figure 8: Long-term stress-strain curves of mortar at $c_1 = 0.38$ with fly ash/cement binder

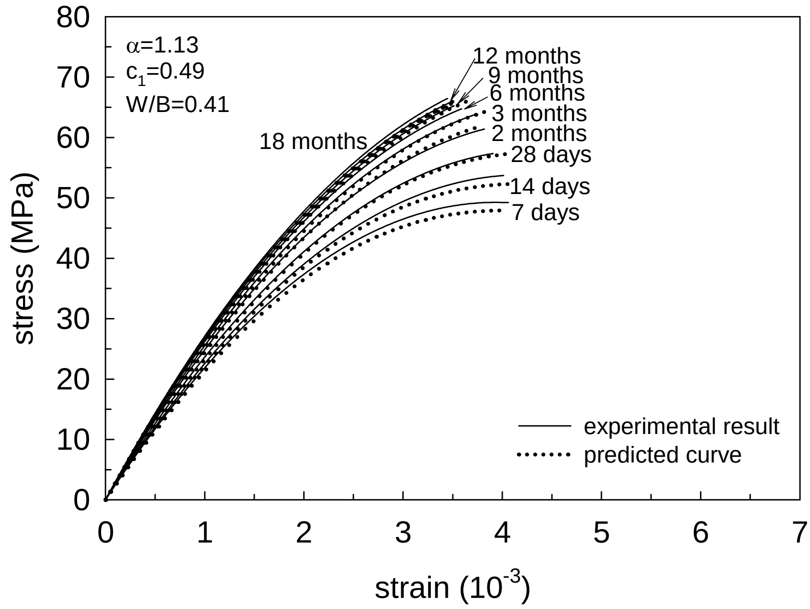

Figure 9: Long-term stress-strain curves of mortar at $c_1 = 0.49$ with fly ash/cement binder

CONCLUDING REMARKS

In this paper we have conducted some experiments to measure the long-term nonlinear stress-strain behavior of cement-based binders, and mortar at three different concentrations of aggregates, and further proposed a micromechanics-based model to predict the overall long-term stress-strain curves of the cement-matrix composite from the properties of the binder and aggregate content. The long-term nonlinear stress-strain curve of the binders was represented by the four-parameter Burgers model, from which its secant Young's modulus can be established as a function of strain. The secant moduli of the matrix are then used in the micromechanics-based composite model to find the overall secant moduli of the composite at a given concentration of inclusions and material age. The overall long-term nonlinear stress-strain relations of the cement-matrix composite then are determined for a given concentration of inclusion. The proposed theory is micromechanics-based, and simple to use. It is also substantiated by experiments, and can pave the way for the study of other concrete characteristics.

ACKNOWLEDGMENTS

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